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Composition Fractional Integral Inequality for the Reiman-Liouville type with applications

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Abstract

In this work, some integral inequalities of fractional order of Riemann-Liouville type are established, which generalized some know inequalities for [8]. These can be used in the analysis of various problems in the theory of certain classes of fractional differential equations and fractional integral equations. Also, some applications for the fractional differential and integral equations are also indicated.

Key Words: Fractional integral inequalities, Holder's inequality

1 Introduction

In the past decades, many authors have studied the fractional integral inequalities and applications via the Riemann- Liouville fractional integral. For example, we refer the reader to [1-10] and the references therein. In[12], F. Qi posed the following inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1}, \; t > 1 . \qquad \text{......} \; (1)$$

Most of the researchers have given considerable attention to (1) and a number of extensions, generalizations and variants have appeared in the literature (see [7, 9, 12, 13, 14]).

Moreover, recently in [8] the author studied the following similar inequality:

$$\int_a^b [f(x)]^{\frac{1}{p}} dt \leq \left[\int_a^b \; f(x)dt\right]^{1-\frac{1}{p}}, \qquad \text{......} \; (2)$$

where $p \geq 2, \; 0 < m \leq f(x) \leq M$ and $M \leq m^{(p-1)^2}/(b-a)^p$ on $[a, b]$.

In the present paper, we establish some fractional inequalities of Riemann-Liouville type for the inequality (1). To illustrate the results, applications of our result to fractional differential and integral equations are also indicated.

2 Definitions of Fractional Integral and Derivatives

In this section, we review a few basic notation and definitions of complex numbers. Moreover, we present some definitions and properties of fractional integrals and derivatives, which we will need later in the analysis of our new results.

Definition 1: [5] We define $L^p(a,b) = L^p[a,b],\ p \geq 1$ as a space of measurable functions for which $\left(\int_a^b |f(x)|^p dx\right)^p < \infty$ and in which the norm is defined as $\|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^p$.

Definition 2: [12, 15] The left Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ is defined by

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} f(\xi) d\xi\,, \quad t \in [a,b],\ Re(\alpha) > 0 \qquad \text{.....}\ (3)$$

where Γ is gamma function.

Definition 3: [12, 15] The right Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ is formally given by

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-\xi)^{\alpha-1} f(\xi) d\xi\,, \quad t \in [a,b],\ Re(\alpha) > 0\,. \qquad \text{......}\ (4)$$

Definition 4: [11] The left Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ is formally defined by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\xi)^{n-\alpha-1} f(\xi) d\xi\,, \quad t \in (a,b),\ n-1 < Re(\alpha) < n \in N$$

$$\text{......}\ (5)$$

Definition 5: [12] The right Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$ is defined as

$$ {}_tD_b^\alpha f(t) = (-1)^n \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (t-\xi)^{n-\alpha-1} f(\xi) d\xi\,, \ t \in (a,b),\ n-1 < Re(\alpha) < n \in N.$$

$$\text{......}\ (6)$$

Proposition 1: [5] Suppose that

(a) $0 < \alpha < 1,\ f \in L^p(a,b),\ g \in L^q(a,b)$. Then
$$\int_a^b f(t) \left({}_aI_t^\alpha g(t)\right) dt = \int_a^b \left({}_tI_b^\alpha f(t)\right) g(t) dt\,, \quad \text{for } p,q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha\,.$$

(b) $0 < Re(\alpha) < 1, f \in {}_tI_b^\alpha(L^p),\ g \in {}_aI_t^\alpha(L^q)$. Then
$$\int_a^b f(t) \left({}_aD_t^\alpha g(t)\right) dt = \int_a^b \left({}_tD_b^\alpha f(t)\right) g(t) dt\,, \quad \text{for } p,q \geq 1 \text{ and } \frac{1}{p} + \frac{1}{q} \leq 1 + \alpha\,.$$

3 Main Results

In this section, we will state our main results and for these we will give their proofs.

Theorem 2: Let $f, g \in L^p(a,b)$ be two nonnegative functions with $0 < m \leq f(t)/\, {}_aI_t^\alpha g(t) \leq M < \infty$ and ${}_aI_t^\alpha f(t),\ {}_tI_b^\alpha g(t) \in L^q(a,b)$ on the interval$[a,b]$. Then for $p,q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1$ we have

$$\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [\,{}_aI_t^\alpha f(t)]^{\frac{1}{q}} [g(t)]^{\frac{1}{p}} dt, \qquad \text{......}\ (7)$$

and hence

$$\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left[\int_a^b [\,{}_aI_t^\alpha f(t)]\right]^{\frac{1}{q}} \left[\int_a^b g(t) dt\right]^{\frac{1}{p}}. \qquad \text{......}\ (8)$$

Proof: Holder's inequality for the functions $f(t)$ and ${}_aI_t^\alpha g(t)$ gives

$$\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt \leq \left[\int_a^b [f(t)dt\right]^{\frac{1}{p}} \left[\int_a^b {}_aI_t^\alpha g(t)\, dt\right]^{\frac{1}{q}}.$$

This leads to

$$\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt \leq \left[\int_a^b [f(t)]^{\frac{1}{p}} [f(t)]^{\frac{1}{q}} dt\right]^{\frac{1}{p}} \left[\int_a^b [\,{}_aI_t^\alpha g(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt\right]^{\frac{1}{q}}.$$

Form the hypotheses we have $[f(t)]^{\frac{1}{p}} \leq M^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{p}}$ and $[\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} \leq m^{-\frac{1}{q}} [f(t)]^{\frac{1}{q}}$ and hence the last equation becomes

$$\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt \leq M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \left[\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt\right]^{\frac{1}{p}} \left[\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt\right]^{\frac{1}{q}}$$

$$= M^{\frac{1}{p^2}} m^{-\frac{1}{q^2}} \int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt. \qquad \text{......} \ (9)$$

Hence, the inequality (7) is proved. Now, we use the following inequality

$\int_a^b [f(t)]^{\frac{1}{p}} [\,{}_aI_t^\alpha g(t)]^{\frac{1}{q}} dt \leq \left[\int_a^b f(t)dt\right]^{\frac{1}{p}} \left[\int_a^b {}_aI_t^\alpha g(t)\right]^{\frac{1}{q}}$ into equation (9) and then using proposition 1, we obtain the inequality(8).

Theorem 3: Let $f \in L^p(a,b)$ be a nonnegative function with $0 < m \leq f(t) \leq M < \infty$ and ${}_aI_t^\alpha f(t) \in L^q(a,b)$ on the interval$[a,b]$. Then for $p \geq 2$ and $M \leq \frac{\Gamma^p(2-\alpha) m^{(p-1)^2}}{(b-a)^{p-\alpha p}}$, we have

$$\int_a^b [f(t)]^{\frac{1}{p}} dt \leq \left[\int_a^b {}_aI_t^\alpha f(t) dt\right]^{1-\frac{1}{p}}. \qquad \text{.....} \ (10)$$

Proof: By substituting $g(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}$ into equation (7) we obtain $\int_a^b [f(t)]^{\frac{1}{p}} dt \leq C \left[\int_a^b {}_aI_t^\alpha f(t) dt]\right]^{1-\frac{1}{p}}$

where

$C = \frac{M^{\frac{1}{p^2}} (b-a)^{(\frac{1-\alpha}{p})}}{\Gamma^{\frac{1}{p}}(2-\alpha) m^{(1-\frac{1}{p})^2}}$. Because $M \leq \frac{\Gamma^p(2-\alpha) m^{(p-1)^2}}{(b-a)^{p-\alpha p}}$, so $C \leq 1$ and this ends the *proof.* If directly, we replace the fractional integral by fractional derivatives in the above Theorem 2, we can proof it similarly.

Example 1: Let $f(x) = (1-t)^2 + 3$ for each $t \in [1\ 2]$ with $m = 3$ and $M = 4$.

By choosing $p = 3$ and $\alpha = 0.5$, we conclude that all the conditions of Theorem 3 are fulfilled and we obtain via a straightforward calculation

$$\int_1^2 ((1-t)^2 + 3)^{\frac{1}{3}} dt = 1.4925 < \left[\int_1^2 {}_1I_t^{0.5} ((1-t)^2 + 3) dt\right]^{\frac{2}{3}} \approx 1.8068.$$

Example 2: Let $f(x) = 4t^3 + 2$ on the interval $[0,1]$ with $m = 2$ and $= 4$.

Taking $p = 4$ and $\alpha = 0.75$, we see that all the conditions of Theorem 3 are fulfilled and a straightforward calculation gives

$$\int_0^1 (4t^3 + 2)^{\frac{1}{2}} dt \approx 1.2519 < \left[\int_0^1 {}_0I_t^{0.75} (4t^3 + 2) dt\right]^{\frac{3}{4}} \approx 1.2842.$$

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